COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED BIPOLAR METRIC SPACES

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Abstract: *In this paper, we establish the existence of coupled fixed point theorems by using weak contractive type, mixed monotone mapping in a bipolar metric space endowed with partial order. Some interesting consequences of our results achieved. Finally, we gave an illustration which presents the applicability of achieved results.*

KEYWORDS: Bipolar metric space, Partial ordering, Weak contractive mapping, Mixed monotone mappings and Coupled fixed point.

1. INTRODUCTION AND PRELIMINARIES

In 1922, S. Banach [1] introduced the concept of Banach contraction principle. It is most celebrated fixed point result in nonlinear analysis. Afterward many investigators established some important fixed point results see ([5]-[10]). Recently, Bhaskar and Lakshmikantham [2], Ran and Reurings [3], Agarwal et al. [4] established some new theorems for contractions in partially ordered metric spaces. The concept of mixed monotone mapping has been introduced by Bhaskar and Lakshmikantham [2] and established some coupled fixed point results for mixed monotone mappings. Subsequently to improve many authors have established coupled fixed point results for mixed monotone see ([11] - [15]). Very recently, in 2016 Mutlu and Gürdal [16] introduced the notion of bipolar metric spaces, which is one of generalizations metric spaces. Also they investigated some fixed point and coupled fixed point results on this space, see ([15], [16]). In this paper, we will continue to study coupled fixed points in the frame of bipolar metric spaces. More squarely, we extend the results of Gnana Bhaskar and Lakshmikantham ([2]) for a mixed monotone contractive mappings. We establish the existence of a ∈ A ∪ B, for a continuous mapping F: (A; B) ⇉ (A; B) such that F (a) = a where (A; B) is a partially ordered set with a bipolar metric on it. In the case that F is not continuous, we prove the existence of a coupled fixed point results by making an additional assumption on (A; B).

Definition 1.1 ([16]): Let A, B be two non-empty sets. Suppose that d: A×B →[0,∞) be a mapping satisfying the below properties:

(i) If d (a, b) = 0, then a=b for all (a, b) ∈ A×B,
(ii) If a = b, then d (a, b) = 0, for all (a, b) ∈ A×B,
(iii) If d (a, b) =d (b, a), for all a, b ∈ A∩B
(iv) If d(a_1, b_2) ≤ d(a_1, b_1) + d(a_2, b_1) +d(a_2, b_2) for all a_1, a_2 ∈ A, and b_1, b_2 ∈B. Then the mapping d is termed as Bipolar-metric of the pair (A, B) and the triple (A, B, d) is termed as Bipolar-metric space.

Example 1.2 ([16]): Let A= (1, ∞) and B= [-1, 1]. Define d: A×B →[0,∞) as d (a, b) =$|a^2 - b^2|$, for all (a, b) ∈ A×B. Then the triple (A, B, d) is a Bipolar-metric space.

Definition 1.3 ([16]): Assume (A_1, B_1) and (A_2, B_2) as two pairs of sets and a function as F: $A_1 \cup B_1 \rightrightarrows A_2 \cup B_2$ is said to be a covariant map. If F (A_1)⊆ A_2 and F (B_1)⊆ B_2, and denote this with F: (A_1, B_1) ⇉ (A_2, B_2,). And the mapping F: $A_1 \cup B_1 \rightleftarrows A_2 \cup B_2$ is said to be a contravariant map. If F (A_1)⊆ B_2, and F (B_1)⊆ A_2, and write F: (A_1, B_1) ⇄ (A_2, B_2). In particular, if d_1 and d_2 are bipolar metric on (A_1, B_1) and (A_2, B_2), respectively, we sometimes use the notation

F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ and F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$.

Definition 1.4 ([16]): Assume (A, B, d) as a bipolar metric space. A point v ∈A ∪B is termed as a left point if v ∈ A, a right point if v ∈ B and a central point if both. Similarly, a sequence $\{a_n\}$ on the set A and a sequence $\{b_n\}$ on the set B are called a left sequence and right sequence respectively. In a bipolar metric space, sequence is the simple term for a left or right sequence. A sequence $\{v_n\}$ is considered convergent to a point v, if and only if $\{v_n\}$ is the left sequence, v is the right point and $\lim_{n\to\infty} d(v_n, v) = 0$; or $\{v_n\}$ is a right sequence, v is a left point and $\lim_{n\to\infty} d(v, v_n) = 0$. A bi-sequence $(\{a_n\}, \{b_n\})$ on (A, B, d) is a sequence on the set A ×B. If the sequence $\{a_n\}$ and $\{b_n\}$ are convergent, then the bi-sequence $(\{a_n\}, \{b_n\})$ is said to be convergent. $(\{a_n\}, \{b_n\})$ is Cauchy sequence, if $\lim_{n\to\infty} d(a_n, b_n) = 0$. In a bipolar metric space, every convergent Cauchy bi-sequence is bi-convergent. A bipolar metric space is called complete, if every Cauchy bisequence is convergent hence biconvergent.

Definition 1.5 ([16]): Let (A_1, B_1, d_1) *and* (A_2, B_2, d_2) be bipolar metric spaces.

(i) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called left-continuous at a point $a_0 \in A_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a_0, b)<\delta$ implies that $d_2(F(a_0), F(b))<\epsilon$ for all b∈ B_1.

(ii) A map F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ is called right-continuous at a point $b_0 \in B_1$, if for every $\epsilon>0$, there is a $\delta>0$ such that $d_1(a, b_0)<\delta$ implies $d_2(F(a), F(b_0))<\epsilon$ for all a ∈ A_1.

(iii) A map F is considered continuous, if it left continuous at each point a ∈ A_1 and righty continuous at each point b∈ B_1.

(iv) A contravariant map F: $(A_1, B_1, d_1) \rightleftarrows (A_2, B_2, d_2)$ is continuous if and only if F: $(A_1, B_1, d_1) \rightrightarrows (A_2, B_2, d_2)$ it is continuous as a covariant map

It is observed from the definition (1.4) that a contravariant or a covariant map F from (A_1, B_1, d_1) to (A_2, B_2, d_2) is continuous if and only if $(u_n) \to v$ on (A_1, B_1, d_1) implies $F((u_n)) \to F(v)$ on (A_2, B_2, d_2).

Definition 1.6: Let (A; B; ≤) be a partial ordered set and F: (A; B) ⇉ (A; B) be a covariant mapping, we say that F is non-decreasing with respect to ≤ if a; b ∈ A ∪B, a ≤ b implies F (a) ≤ F (b), and similarly, a non-increasing mapping is defined.

Definition 1.7: Let (A; B; ≤) be a partially ordered set and F: (A^2 ; B^2) ⇉ (A; B) be a covariant map. The map F has the mixed monotone property, if F(a; b) is monotone non-decreasing in a and is monotone non-increasing in b, that is, for any *(a;b)*∈ $A^2 \cup B^2$,

$(a_1, a_2) \in A^2$; $a_1 \le a_2 \Rightarrow F(a_1; b) \le F(a_2; b)$.

$(b_1, b_2) \in B^2$; $b_1 \le b_2 \Rightarrow F(a, b_1) \ge F(a, b_2)$.

Definition 1.8. Let F: $(A^2;B^2) \rightrightarrows$ (A; B) be a covariant map, an element (a; b) ∈ $A^2 \cup B^2$ is called coupled fixed point of F if F (a; b) = a; and F (b; a) = b

2. MAIN RESULTS

Let (A; B; ≤) be a partially ordered set and d be a bipolar metric on (A; B) such that (A; B; d, ≤) is complete bipolar metric space. Moreover, we endow the product space (A^2 ; B^2) with the following partial order: For (a; b), (p; q)∈ $A^2 \cup B^2$ (p; q) ≤ (a; b) ⇔ a≥p; b≤q. We begin with the following theorem that achieves the existence of a fixed point results for a mapping F on the product space (A^2; B^2).

Theorem 2.1: Let F: $(A^2;B^2) \rightrightarrows$ (A; B) be a covariant map. If F is a continuous mapping having the mixed monotone property on (A; B) and μ, λ be a non –negative constants with the condition d (F (l; m); F (r; s)) ≤ μd (l; r) +λ d (m; s) for all l; m ∈ A and r; s ∈ B with l ≥ r; m ≤ s; **(1)** and μ + λ < 1. If there is $(l_0 ; m_0) \in A^2 \cup B^2$ such that $l_0 \le F(l_0 ; m_0)$, $m_0 \ge (m_0, l_0)$. Then there exist (l; m) ∈ $A^2 \cup B^2$ such that the mapping F: $A^2 \cup B^2 \to$ A ∪ B has F (l; m) = l; and F (m; l) = m

Proof: Let $l_0; m_0 \in$ A and $r_0; s_0 \in$ B, choose an elements $l_1 ; m_1 \in$ A and $r_1 ; s_1 \in$ B,such that $l_0 \le F(l_0; m_0) = l_1$; $m_0 \ge F(m_0, l_0) = m_1$;

And also $r_0 \le F(r_0;s_0) = r_1$, $s_0 \ge F(s_0, r_0) = s_1$,

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Similarly, we take $F(l_1; m_1) = l_2$ $F(m_1, l_1) = m_2$; and also $F(r_1; s_1) = r_2$ $F(s_1, r_1;) = s_2$: Denote

$F^2(l_0, m_0)$= F(F(l_0, m_0) , F(m_0, l_0))

=F(l_1, m_1)=l_2

$F^2(m_0, l_0)$=F(F(m_0, l_0) , F(l_0, m_0))

=F (m_1, l_1)=m_2

$F^2(r_0, s_0)$=F(F(r_0, s_0) , F(s_0, r_0))

=F (r_1, s_1)=r_2

$F^2(s_0, r_0)$=F(F(s_0, r_0) , F(r_0, s_0))

=F (s_1, r_1)=s_2

In this process, we get a bi-sequences $(F^n(l_0, m_0), F^n(m_0, l_0)) = (l_n, m_n)$ and

$(F^n(r_0, s_0), F^n(s_0, r_0)) = (r_n, s_n)$ with

$l_{n+1} = F^{n+1}(l_0, m_0)$

= F ($F^n(l_0, m_0), F^n(m_0, l_0)$)=F($l_n, m_n$)

$m_{n+1} = F^{n+1}(m_0, l_0)$

= F ($F^n(m_0, l_0), F^n(l_0, m_0)$)=F($m_n, l_n$)

$r_{n+1} = F^{n+1}(r_0, s_0)$

= F ($F^n(r_0, s_0), F^n(s_0, r_0)$)=F($r_n, s_n$)

$s_{n+1} = F^{n+1}(s_0, r_0)$

= F ($F^n(s_0, r_0), F^n(r_0, s_0)$)=F($s_n, r_n$) $\forall n \in N$

Obviously, verify that

$l_0 \leq F(l_0, m_0) = l_1 \leq F^2(l_0, m_0) = l_2 \leq \ldots\ldots \leq F^{n+1}(l_0, m_0) \leq \ldots\ldots$

$m_0 \geq F(m_0, l_0) = m_1 \geq F^2(m_0, l_0) = m_2 \geq \ldots\ldots \geq F^{n+1}(m_0, l_0) \geq \ldots\ldots$

$r_0 \leq F(r_0, s_0) = r_1 \leq F^2(r_0, s_0) = r_2 \leq \ldots\ldots \leq F^{n+1}(r_0, s_0) \leq \ldots\ldots$

$s_0 \geq F(s_0, r_0) = s_1 \geq F^2(s_0, r_0) = s_2 \geq \ldots\ldots \geq F^{n+1}(s_0, r_0) \geq \ldots\ldots$

Now, Show that, for n∈ N and let $\mu + \lambda = \xi$

$$d(F^n(l_0, m_0), F^{n+1}(r_0, s_0)) + d(F^n(m_0, l_0), F^{n+1}(s_0, r_0)) \leq \xi^n \left[d\left(l_0, F(r_0, s_0)\right) + d\left(m_0, F(s_0, r_0)\right)\right] \quad \textbf{(2)}$$

Indeed, for n=1, using $F(l_0, m_0) \geq l_0$, $F(m_0, l_0) \leq m_0$ and $F(r_0, s_0) \geq r_0$ $F(s_0, r_0) \leq s_0$

$$d(F(l_0, m_0), F^2(r_0, s_0)) = d(F(l_0, m_0), F(F(r_0, s_0), F(s_0, r_0))) \leq \mu\, d(l_0, F(r_0, s_0)) + \lambda\, d(m_0, F(s_0, r_0)) \quad \textbf{(3)}$$

and

$$d(F(m_0, l_0), F^2(s_0, r_0)) = d(F(m_0, l_0), F(F(s_0, r_0), F(r_0, s_0))) \leq \mu\, d(m_0, F(s_0, r_0)) + \lambda\, d(l_0, F(r_0, s_0)) \quad \textbf{(4)}$$

Combing **(3)** and **(4)** we have

$$d(F(l_0, m_0), F^2(r_0, s_0)) + d(F(m_0, l_0), F^2(s_0, r_0)) \leq (\mu+\lambda)\begin{bmatrix} d\left(l_0, F(r_0, s_0)\right) \\ + d(m_0, F(s_0, r_0)) \end{bmatrix} \leq \xi \begin{bmatrix} d\left(l_0, F(r_0, s_0)\right) \\ + d(m_0, F(s_0, r_0)) \end{bmatrix}$$

And also show that

$$d(F^{n+1}(l_0, m_0), F^n(r_0, s_0)) + d(F^{n+1}(m_0, l_0), F^n(s_0, r_0)) \leq \xi^n \begin{bmatrix} d(F(l_0, m_0), r_0) \\ + d(F(m_0, l_0), s_0) \end{bmatrix} \quad \textbf{(5)}$$

Indeed, for n=1, using $F(l_0, m_0) \geq l_0$, $F(m_0, l_0) \leq m_0$ and $F(r_0, s_0) \geq r_0$, $F(s_0, r_0) \leq s_0$

$$d(F^2(l_0, m_0), F(r_0, s_0)) = d(F(F(l_0, m_0), F(m_0, l_0)), F(r_0, s_0)) \leq \mu\, d(F(l_0, m_0), r_0) + \lambda\, d(F(m_0, l_0)), s_0) \quad \textbf{(6)}$$

d($F^2(m_0, l_0)$, F(s_0, r_0))

= d(F(F (m_0, l_0), F (l_0, m_0),) F(s_0, r_0))
≤ μ d (F (m_0, l_0), s_0)+λ d (F (l_0, m_0), r_0) **(7)**

Combining **(6)** and **(7)**

d ($F^2(l_0, m_0)$, F(r_0, s_0))

+ d(F^2 (m_0, l_0), F(s_0, r_0)) ≤ (μ+λ) $\begin{bmatrix} \text{d (F }(l_0, m_0),\ r_0) \\ +\text{ d (F }(m_0, l_0),\ s_0) \end{bmatrix}$

≤ξ $\begin{bmatrix} \text{d (F }(l_0, m_0),\ r_0) \\ +\text{ d (F }(m_0, l_0),\ s_0) \end{bmatrix}$

Assume that **(2)** and **(5)** hold. Using

$F^{n+1}(l_0, m_0) \geq F^n(l_0, m_0)$, $F^{n+1}(m_0, l_0) \leq F^n(m_0, l_0)$ and $F^{n+1}(r_0, s_0) \geq F^n(r_0, s_0)$,

$F^{n+1}(s_0, r_0) \leq F^n(s_0, r_0)$.

Moreover,

$d(F^n(l_0, m_0), F^n(r_0, s_0))$

$= d(F(F^{n-1}(l_0, m_0), F^{n-1}(m_0, l_0)),$

$F(F^{n-1}(r_0, s_0), F^{n-1}(s_0, r_0))$

≤μ d($F^{n-1}(l_0, m_0), F^{n-1}(r_0, s_0)$)

+λ d($F^{n-1}(m_0, l_0), F^{n-1}(s_0, r_0)$)

(8)

and

$d(F^n(m_0, l_0), F^n(s_0, r_0))$

$= d(F(F^{n-1}(m_0, l_0), F^{n-1}(l_0, m_0)),$

$F(F^{n-1}(s_0, r_0), F^{n-1}(r_0, s_0))$

≤μ d($F^{n-1}(m_0, l_0), F^{n-1}(s_0, r_0)$)

+λ d($F^{n-1}(l_0, m_0), F^{n-1}(r_0, s_0)$)

(9)

For all n∈N Combining **(8)** and **(9),** then

$d(F^n(l_0, m_0), F^n(r_0, s_0))$

$+ d(F^n(m_0, l_0), F^n(s_0, r_0))$

≤ (μ+λ) $\begin{bmatrix} \text{d}(F^{n-1}(l_0, m_0), F^{n-1}(r_0, s_0)) \\ +\text{ d}(F^{n-1}(m_0, l_0), F^{n-1}(s_0, r_0)) \end{bmatrix}$

$\leq \xi \begin{bmatrix} \text{d}(F^{n-1}(l_0, m_0), F^{n-1}(r_0, s_0)) \\ +\text{ d}(F^{n-1}(m_0, l_0), F^{n-1}(s_0, r_0)) \end{bmatrix}$

⋮

$\leq \xi \begin{bmatrix} \text{d}(F(l_0, m_0),\ F(r_0, s_0)) \\ +\text{ d}(F(m_0, l_0),\ F(s_0, r_0)) \end{bmatrix}$

$\leq \xi \begin{bmatrix} \text{d}(F(l_0, m_0),\ r_0) \\ +\text{ d}(F(m_0, l_0),\ s_0) \end{bmatrix}$ **(10)**

Using the property (B_4), we get that

$d(F^n(l_0, m_0), F^m(r_0, s_0))$

$\leq d(F^n(l_0, m_0), F^{n+1}(r_0, s_0))$

$+d(F^{n+1}(l_0, m_0), F^{n+1}(r_0, s_0))$

+……… $+ d(F^{m-1}(l_0, m_0), F^m(r_0, s_0))$

and

$d(F^n(m_0, l_0), F^m(s_0, r_0))$

$\leq d(F^n(m_0, l_0), F^{n+1}(s_0, r_0))$

$+ d(F^{n+1}(m_0, l_0), F^{n+1}(s_0, r_0))$

+……… $+ d(F^{m-1}(m_0, l_0), F^m(s_0, r_0))$ **(11)**

Also, we have

$d(F^m(l_0, m_0), F^n(r_0, s_0))$

$\leq d(F^m(l_0, m_0), F^{m-1}(r_0, s_0))$

$+ d(F^{m-1}(l_0, m_0), F^{m-1}(r_0, s_0))$

+……… $+ d(F^{n+1}(l_0, m_0), F^n(r_0, s_0))$

and

$d(F^m(m_0, l_0), F^n(s_0, r_0))$

$\leq d(F^m(m_0, l_0), F^{m-1}(s_0, r_0))$

$+ d(F^{m-1}(m_0, l_0), F^{m-1}(s_0, r_0))$

+……… $+ d(F^{n+1}(m_0, l_0), F^n(s_0, r_0))$ **(12)**

For each n, m ∈ N, n<m. From **(2), (5), (10), (11)** and **(12)**, then we get

$d(F^n(l_0, m_0), F^m(r_0, s_0))$

$+ d(F^n(m_0, l_0), F^m(s_0, r_0))$

$\leq d(F^n(l_0, m_0), F^{n+1}(s_0, r_0))$

$+ d(F^n(m_0, l_0), F^{n+1}(s_0, r_0))$

$+ d(F^{n+1}(l_0, m_0), F^{n+1}(s_0, r_0))$

$+ d(F^{n+1}(m_0, l_0), F^{n+1}(s_0, r_0)) + \cdots\cdots\cdots$

$+ d(F^{m-1}(l_0, m_0), F^m(s_0, r_0))$

$+ d(F^{m-1}(m_0, l_0), F^m(s_0, r_0))$

$\leq (\xi^n + \xi^{n+1} + \cdots\cdots\cdots + \xi^{m-1})$

$(d(l_0, F(r_0, s_0)) + d(m_0, F^m(s_0, r_0)))$

$$\leq \frac{\xi^n}{1-\xi}\begin{bmatrix} d(l_0, F(r_0, s_0)) \\ + d(m_0, F^m(s_0, r_0)) \end{bmatrix} \quad \textbf{(13)}$$

And

$d(F^m(l_0, m_0), F^n(r_0, s_0))$

$+ d(F^m(m_0, l_0), F^n(s_0, r_0))$

$\leq d(F^m(l_0, m_0), F^{m-1}(s_0, r_0))$

$+ d(F^m(m_0, l_0), F^{m-1}(s_0, r_0))$

$+ d(F^{m-1}(l_0, m_0), F^{m-1}(s_0, r_0))$

$+ d(F^{m-1}(m_0, l_0), F^{m-1}(s_0, r_0))$

$+ \cdots\cdots\cdots + d(F^{n+1}(l_0, m_0), F^n(r_0, s_0))$

$+ d(F^{n+1}(m_0, l_0), F^n(s_0, r_0))$

$\leq (\xi^m + \xi^{m-1} + \cdots\cdots\cdots + \xi^n)$

$[d(F(l_0, m_0), r_0) + d(F(m_0, l_0), s_0)]$

$$\leq \frac{\xi^n}{1-\xi}\begin{bmatrix} d(F(l_0, m_0), r_0) \\ + d(F(m_0, l_0), s_0) \end{bmatrix} \quad \textbf{(14)}$$

For n< m. Since, for an arbitrary $\epsilon > 0$, there exists n_0 such that $\frac{\xi^n}{1-\xi} < \frac{\epsilon}{3}$.

From (13) and (14) , we get

$$[d(F^n(l_0, m_0), F^m(r_0, s_0)) + d(F^n(m_0, l_0), F^m(s_0, r_0))] < \frac{\epsilon}{3}$$

For n, m ≥ n_0. Then

$(\{F^n(l_0, m_0)\}, \{F^m(r_0, s_0)\})$ and

$(\{F^n(m_0, l_0)\}, \{F^m(s_0, r_0)\})$ are Cauchy bi-sequence in (A, B). Since (A, B, d) is a complete bipolar metric spaces, there exists l, m ∈ A and r, s∈B such that

$\lim_{n\to\infty} F^n(l_0, m_0)$ = r, $\lim_{n\to\infty} F^n(m_0, l_0)$ = s,

and $\lim_{n\to\infty} F^n(s_0, r_0)$ = m, $\lim_{n\to\infty} F^n(r_0, s_0)$ = l,

(15)

First we show that F (l, m) = r, F (m, l)= s and F(r, s)=l, F(s, r)=m.

Let $\epsilon > 0$. Since F is continuous at (l, m), for given $\frac{\epsilon}{3} > 0$, there exist $\delta > 0$ such that

d(l, r) +d(m, s) $< \delta$ implies that

d(F(l, m), F(r, s)) $< \frac{\epsilon}{3}$

Since $\{F^n(l_0, m_0)\} \to r$, $\{F^n(m_0, l_0)\} \to s$ and $\{F^n(r_0, s_0)\} \to l$, $\{F^n(s_0, r_0)\} \to m$,

For $\boldsymbol{\eta}$= min $\{\frac{\epsilon}{3}, \frac{\delta}{3}\}$, then there exists n_1 ∈N with

d$(F^n(l_0, m_0), r) < \boldsymbol{\eta}$, d$(F^n(m_0, l_0), s) < \boldsymbol{\eta}$, and

d$(F^n(r_0, s_0), l) < \boldsymbol{\eta}$, d$(F^n(s_0, r_0), m) < \boldsymbol{\eta}$

for all n $\geq n_1$ and every $\boldsymbol{\eta} > 0$, since $(\{F^n(l_0, m_0)\}, \{F^n(r_0, s_0)\})$ and $(\{F^n(m_0, l_0)\}, \{F^n(s_0, r_0)\})$ are Cauchy sequences. We get

d($F^n(l_0, m_0)$, $F^n(r_0, s_0)$) $< \boldsymbol{\eta}$ and d$(F^n(m_0, l_0)$, $F^n(s_0, r_0)) < \boldsymbol{\eta}$**.**

So from (iv) in definition 1.1, we get

d(F(l, m), r)≤ d(F(l, m), $F^{n+1}(r_0, s_0)$)

+ d$(F^{n+1}(l_0, m_0), F^{n+1}(r_0, s_0))$

+d $(F^{n+1}(l_0, m_0), r)$

≤d(F(l, m), F$(F^n(r_0, s_0), F^n(s_0, r_0))$

$+ \text{d}(\text{F}(F^n(l_0, m_0), F^n(m_0, l_0)),$

$\text{F}(F^n(r_0, s_0), F^n(s_0, r_0)))$

$+\text{d}(F^{n+1}(l_0, m_0), r)$

$\leq \frac{\epsilon}{3} + \boldsymbol{\eta} + \boldsymbol{\eta} < \epsilon$

For each n∈N. This implies d(F(l, m), r)=0. Hence F(l, m)=r . Similarly, we can prove that F (m, l)= s and F(r, s)=l, F(s, r)=m. On the other hand,

$\text{d(l, r)} = \text{d}(\lim_{n\to\infty} F^n(l_0, m_0), \lim_{n\to\infty} F^n(r_0, s_0))$

$= \lim_{n\to\infty} \text{d}(F^n(l_0, m_0), F^n(r_0, s_0)) = 0$

and

$\text{d(m, s)} = \text{d}(\lim_{n\to\infty} F^n(m_0, l_0), \lim_{n\to\infty} F^n(s_0, r_0))$

$= \lim_{n\to\infty} \text{d}(F^n(m_0, l_0), F^n(s_0, r_0)) = 0.$

Therefore, l= r and m= s and hence

F(l , m)=l and F(m, l)= m.

The achieved Theorem is still valid for the covariant map F is not necessarily continuous. Instead, we require that underlying bipolar metric space (A, B) has an additional postulate. We discuss this in the following result.

Theorem 2.2. Let (A, B, ≤) be a partially ordered set and suppose that (A, B,d, ≤) is complete bipolar metric spaces on (A, B) such that (A, B) has the following postulate:

(i) If a non-decreasing sequence $(\{l_n\}, \{m_n\}) \to l$ then $(l_n, m_n) \leq l$ ∀n
(ii) If a non-decreasing sequence $(\{m_n\}, \{l_n\}) \to m$ then $m \leq (m_n, l_n)$ ∀n

Let F: $(A^2;B^2) \rightrightarrows (A; B)$ be a covariant mapping having the mixed monotone property on (A, B) and μ, λ be a non –negative constants with the condition
d (F (l; m); F (r; s)) ≤ μd (l; r) +λ d (m; s) *for all* l; m ∈ A and r; s ∈ B with l ≥ r; m ≤ s; **(16)** and μ + λ < 1. If there is $(l_0 ; m_0) \in A^2 \cup B^2$ such that

$l_0 \leq F(l_0, m_0)$, $m_0 \geq F(m_0, l_0)$. Then there exist (l; m) ∈ $A^2 \cup B^2$ such that the mapping F: $A^2 \cup B^2 \to A \cup B$ has F (l; m) = l; and F (m; l) = m.

Proof: Following the proof of previous Theorem 2.1, we only have to prove that

F(l, m)= l and F(r, s)= r, let $\epsilon > 0$. Since $\{F^n(l_0, m_0)\} \to r$, $\{F^n(m_0, l_0)\} \to s$ and $\{F^n(r_0, s_0)\} \to l$, $\{F^n(s_0, r_0)\} \to m$, then there exists $n_1 \in$ N such that for all n $\geq n_1$ and every $\epsilon > 0$, we have

$\text{d}(F^n(l_0, m_0), r) < \frac{\epsilon}{3}$, $\text{d}(F^n(m_0, l_0), s) < \frac{\epsilon}{3}$, and

$\text{d}(F^n(r_0, s_0), l) < \frac{\epsilon}{3}$, $\text{d}(F^n(s_0, r_0), m) < \frac{\epsilon}{3}$.

For all n $\geq n_1$ and every $\epsilon > 0$, since $(\{F^n(l_0, m_0)\}, \{F^n(r_0, s_0)\})$ and $(\{F^n(m_0, l_0)\}, \{F^n(s_0, r_0)\})$ are Cauchy sequences. We get

$\text{d}(F^n(l_0, m_0), \ F^n(r_0, s_0)) < \frac{\epsilon}{3}$

and $\text{d}(F^n(m_0, l_0), \ F^n(s_0, r_0)) < \frac{\epsilon}{3}$

Taking n_1 ∈N with for all n≥ n_1 and using

$F^n(l_0, m_0) \leq$ r, $F^n(s_0, r_0) \geq$ m and $F^{n+1}(r_0, s_0) \leq$ l, $F^{n+1}(m_0, l_0) \geq$s.

We obtain

$\text{d}(\text{F(l, m), r}) \leq \text{d}(\text{F(l, m)}, F^{n+1}(r_0, s_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), F^{n+1}(r_0, s_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), \text{r})$

$\leq \text{d}(\text{F(l, m)}, F(F^n(r_0, s_0), F^n(s_0, r_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), F^{n+1}(r_0, s_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), \text{r})$

$\leq \mu\, \text{d}(\text{l}, F^n(r_0, s_0)) + \lambda\, \text{d}(\text{m}, F^n(s_0, r_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), F^{n+1}(r_0, s_0))$

$+ \text{d}(F^{n+1}(l_0, m_0), \text{r})$

$< \mu\frac{\epsilon}{3} + \lambda\frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} < (\mu+\lambda)\frac{\epsilon}{3} + 2\frac{\epsilon}{3} < \xi\frac{\epsilon}{3} + 2\frac{\epsilon}{3} < \epsilon$

This implies that d(F(l, m), r)=0, hence

F(l, m)= r. Similarly, we obtain F(m, l) =s, F(r, s) =l and F(s, r)=m. On the other hand,

$d(l, r) = d(\lim_{n\to\infty} F^n(l_0, m_0), \lim_{n\to\infty} F^n(r_0, s_0))$

$= \lim_{n\to\infty} d(F^n(l_0, m_0), F^n(r_0, s_0)) = 0$

and

$d(m, s) = d(\lim_{n\to\infty} F^n(m_0, l_0), \lim_{n\to\infty} F^n(s_0, r_0))$

$= \lim_{n\to\infty} d(F^n(m_0, l_0), F^n(s_0, r_0)) = 0.$

Therefore, l= r and m= s and hence F(l , m)=l and F(m, l)= m.

Further, we show that the coupled fixed point is unique, in fact to provided that the space $(A^2;B^2)$ endowed with the partial order having the every pair of elements has either a lower bound or an upper bound. That is for every (l, m), $(l^*, m^*) \in A^2 \cup B^2$, there is an element (p, q) $\in A^2 \cup B^2$ such that it is comparable to (l, m) and (l^*, m^*) **(17)**

Theorem 2.3: Adding condition (17) to the hypothesis of Theorem 2.2, then the mapping $F: A^2 \cup B^2 \rightarrow A \cup B$ has unique coupled fixed point.

Proof: Let $(l^*, m^*) \in A^2 \cup B^2$ be a another fixed point of F. Then we prove that

$d(l, l^*) + d(m, m^*) = 0$, where

$\lim_{n\to\infty} F^n(l_0, m_0) = l$ and $\lim_{n\to\infty} F^n(m_0, l_0) = m$.

If $(l^*, m^*) \in A^2$ and (l, m) is comparable to (l^*, m^*) with respect to the partial ordering in $(A^2;B^2)$, then for every n∈N we have $(F^n(l, m), F^n(m, l)) = (l, m)$ is comparable to $(F^n(l^*, m^*), F^n(m^*, l^*))$.

Now $d(l^*, l) = d(F^n(l^*, m^*), F^n(l, m))$

$$= d\begin{pmatrix} F(F^{n-1}(l^*, m^*), F^{n-1}(m^*, l^*)), \\ F(F^{n-1}(l, m), F^{n-1}(m, l) \end{pmatrix}$$

$$\leq \mu d(F^{n-1}(l^*, m^*), F^{n-1}(l, m)) + \lambda d(F^{n-1}(m^*, l^*), F^{n-1}(m, l)) \quad \textbf{(18)}$$

And

$d(m^*, m) = d(F^n(m^*, l^*), F^n(m, l))$

$$= d\begin{pmatrix} F(F^{n-1}(m^*, l^*), F^{n-1}(l^*, m^*), \\ F(F^{n-1}(m, l), F^{n-1}(l, m)) \end{pmatrix}$$

$$\leq \mu d(F^{n-1}(m^*, l^*), F^{n-1}(m, l)) + \lambda d(F^{n-1}(l^*, m^*), F^{n-1}(l, m)) \quad \textbf{(19)}$$

For all n∈N, combining (18) and (19)

$d(l^*, l) + d(m^*, m)$

$$\leq (\mu+\lambda)(d(F^{n-1}(l^*, m^*), F^{n-1}(l, m))) + (\mu+\lambda)(d(F^{n-1}(m^*, l^*), F^{n-1}(m, l)))$$

$$\leq \xi \begin{bmatrix} d(F^{n-1}(l^*, m^*), F^{n-1}(l, m)) \\ +d(F^{n-1}(m^*, l^*), F^{n-1}(m, l)) \end{bmatrix}$$

⋮

$$\leq \xi^n \begin{pmatrix} d(F(l^*, m^*), F(l, m)) \\ +d(F(m^*, l^*), F(m, l)) \end{pmatrix}$$

$$\leq \xi^n (d(l^*, l) + d(m^*, m))$$

Since ξ <1 which implies

$d(l^*, l) + d(m^*, m) = 0$. Hence we obtain $l = l^*$ and $m = m^*$.

Similarly, if If $(l^*, m^*) \in B^2$ and (l, m) is comparable to (l^*, m^*) with respect to the partial ordering in $(A^2;B^2)$, then we have $l = l^*$ and $m = m^*$.

If $(l^*, m^*) \in A^2$ and (l, m) is not comparable to (l^*, m^*) then there exist two comparable lower or upper bounds (a, b), $(a^*, b^*) \in A^2 \cup B^2$ of (l, m) and (l^*, m^*). Then for all n∈N,

$(F^n(a, b), F^n(b, a)) = (a, b)$ and $(F^n(a^*, b^*), F^n(b^*, a^*)) = (a^*, b^*)$ is comparable to $(F^n(l, m), F^n(m, l)) = (l, m)$ and $(F^n(l^*, m^*), F^n(m^*, l^*)) = (l^*, m^*)$

Now $d(l^*, l) = d(F^n(l^*, m^*), F^n(l, m))$

$$\leq d(F^n(l^*, m^*), F^n(a, b)) + d(F^n(a^*, b^*), F^n(a, b)) + d(F^n(a^*, b^*), F^n(l, m))$$

$$\leq \mu \begin{bmatrix} d(F^{n-1}(l^*, m^*), F^{n-1}(a, b)) \\ +d(F^{n-1}(a^*, b^*), F^{n-1}(a, b)) \\ +d(F^{n-1}(a^*, b^*), F^{n-1}(l, m)) \end{bmatrix} + \lambda \begin{bmatrix} d(F^{n-1}(m^*, l^*), F^{n-1}(b, a)) \\ +d(F^{n-1}(b^*, a^*), F^{n-1}(b, a)) \\ +d(F^{n-1}(b^*, a), F^{n-1}(m, l)) \end{bmatrix}$$

(20)

And

$d(m^*, m) = d(F^n(m^*, l^*), F^n(m, l))$

$\leq d(F^n(m^*, l^*), F^n(b, a))$

$+d(F^n(b^*, a^*), F^n(b, a))$

$+d(F^n(b^*, a^*), F^n(m, l))$

$$\leq \lambda \begin{bmatrix} d\left(F^{n-1}(l^*, m^*), F^{n-1}(a, b)\right) \\ +d\left(F^{n-1}(a^*, b^*), F^{n-1}(a, b)\right) \\ +d(F^{n-1}(a^*, b^*), F^{n-1}(l, m)) \end{bmatrix}$$

$$+\mu \begin{bmatrix} d\left(F^{n-1}(m^*, l^*), F^{n-1}(b, a)\right) \\ +d\left(F^{n-1}(b^*, a^*), F^{n-1}(b, a)\right) \\ +d(F^{n-1}(b^*, a), F^{n-1}(m, l)) \end{bmatrix}$$

(21)

For all n∈N combining (20) and (21), we get

$d(l^*, l) + d(m^*, m)$

$$\leq (\mu+\lambda) \begin{bmatrix} d\left(F^{n-1}(l^*, m^*), F^{n-1}(a, b)\right) \\ +d\left(F^{n-1}(a^*, b^*), F^{n-1}(a, b)\right) \\ +d(F^{n-1}(a^*, b^*), F^{n-1}(l, m)) \end{bmatrix}$$

$$+(\mu+\lambda) \begin{bmatrix} d\left(F^{n-1}(m^*, l^*), F^{n-1}(b, a)\right) \\ +d\left(F^{n-1}(b^*, a^*), F^{n-1}(b, a)\right) \\ +d(F^{n-1}(b^*, a), F^{n-1}(m, l)) \end{bmatrix}$$

$$\leq \xi \left\{ \begin{matrix} \begin{bmatrix} d\left(F^{n-1}(l^*, m^*), F^{n-1}(a, b)\right) \\ +d\left(F^{n-1}(a^*, b^*), F^{n-1}(a, b)\right) \\ +d(F^{n-1}(a^*, b^*), F^{n-1}(l, m)) \end{bmatrix} \\ + \begin{bmatrix} d\left(F^{n-1}(m^*, l^*), F^{n-1}(b, a)\right) \\ +d\left(F^{n-1}(b^*, a^*), F^{n-1}(b, a)\right) \\ +d(F^{n-1}(b^*, a), F^{n-1}(m, l)) \end{bmatrix} \end{matrix} \right\}$$

⋮

$$\leq \xi^n \left\{ \begin{matrix} \begin{bmatrix} d\left(F(l^*, m^*), F(a, b)\right) \\ +d\left(F(a^*, b^*), F(a, b)\right) \\ +d(F(a^*, b^*), F(l, m)) \end{bmatrix} \\ + \begin{bmatrix} d\left(F(m^*, l^*), F(b, a)\right) \\ +d\left(F(b^*, a^*), F(b, a)\right) \\ +d(F(b^*, a), F(m, l)) \end{bmatrix} \end{matrix} \right\}$$

$$\leq \xi^n \begin{bmatrix} d(l^*, a) + d(a^*, a) \\ +d(a^*, l) + d(m^*, b) \\ +d(b^*, b) + d(b^*, m) \end{bmatrix}$$

$\to 0$ as $n \to \infty$

So that $d(l^*, l) + d(m^*, m)=0$ implies $l^*=l$ and $m^*=m$. Similarly, if $(l^*, m^*) \in B^2$ and (l, m) is incomparable to (l^*, m^*) with respect to the partial ordering in $(A^2;B^2)$, then we have $l=l^*$ and $m=m^*$. Hence (l, m) is unique coupled fixed point of F.

If we take equal constants μ and λ in Theorem 2.1, then following corollary is obtained.

Corollary 1: Let F: $(A^2; B^2) \rightrightarrows (A; B)$ be a covariant map. If F is a continuous mapping having the mixed monotone property on (A; B) and μ ∈ [0, 1) with the condition $d(F(l; m); F(r; s)) \leq \frac{\mu}{2}(d(l; r) + d(m; s))$ for all l; m ∈ A and r; s ∈ B with l ≥ r; m ≤ s **(22)** If there exist $(l_0; m_0) \in A^2 \cup B^2$ such that $l_0 \leq F(l_0; m_0)$, $m_0 \geq F(m_0, l_0)$. Then there exist (l; m) ∈ $A^2 \cup B^2$ such that the mapping F: $A^2 \cup B^2 \to A \cup B$ has F (l; m) = l; and F (m; l) = m.

Corollary 2: Corollary 1 satisfy to the hypothesis of Theorem 2.1, Theorem 2.2 and Theorem 2.3. Then F: $A^2 \cup B^2 \to A \cup B$ has a unique coupled fixed point.

Definition 2.4: Let

F: $(A\times B; B\times A) \rightrightarrows (A; B)$ be a covariant map, an element (a; p) ∈ $A\times B$ is called coupled fixed point of F if F (a; p) = a; and F (p; a) = p.

Theorem 2.5: Let

F: $(A\times B; B\times A) \rightrightarrows (A; B)$ be a covariant map. If F is a continuous mapping having the mixed monotone property on (A, B) and μ, λ be a non – negative constants with the condition $d(F(l; r); F(s; m)) \leq \mu d(l; s) + \lambda d(m; r)$ for all l; m ∈ A and r; s ∈ B with l ≥ s; m ≤ r; **(23)** and μ + λ < 1. If there exist

$(l_0; m_0) \in (A\times B) \cup (B\times A)$ such that $l_0 \leq F(l_0; r_0)$, $r_0 \geq F(r_0, l_0)$. Then there exist (l; m) ∈ $(A\times B) \cup (B\times A)$ such that the mapping F: $(A\times B) \cup (B\times A) \to A \cup B$ has F (l; r) = l; and F (r; l) = r

Theorem 2.6. Let (A, B, ≤) be a partially ordered set and suppose that (A, B,d, ≤) is complete bipolar metric spaces on (A, B) such that (A, B) has the following postulate:

(i) If a non-decreasing sequence $(\{l_n\}, \{r_n\}) \to$ l then $(l_n, r_n) \leq$ l ∀n
(ii) If a non-decreasing sequence $(\{r_n\}, \{l_n\}) \to$ r then $r \leq (r_n, l_n)$ ∀n

Let F: $(A \times B,\ B \times A) \rightrightarrows$ (A; B) be a covariant mapping having the mixed monotone property on (A, B) and μ, λ be a non –negative constants with satisfying the condition of covariant mapping d (F (l; r); F (s; m)) ≤ μd (l; s) +λ d (m; r) for all l; m ∈ A and r; s ∈ B with l ≥ s; m ≤ r; **(24)** and μ + λ < 1. If $(l_0 ; r_0) \in (A \times B) \cup (B \times A)$

Such that $l_0 \leq F(l_0 ; r_0)$, $r_0 \geq F(r_0, l_0)$. Then there exist (l; r) ∈ $(A \times B) \cup (B \times A)$ such that

F:$(A \times B) \cup (B \times A) \to A \cup B$ has

F (l; r) = l; and F (r; l) = r.

Corollary 3: Let

F: $(A \times B,\ B \times A) \rightrightarrows$ (A; B) be a covariant map. If F is a continuous mapping having the mixed monotone property on (A; B) and μ ∈ [0, 1) with satisfying the condition of covariant mapping d (F (l; r); F (s; m)) $\leq \frac{\mu}{2}\begin{pmatrix} d(l;\ s) \\ + d(m;\ r)\end{pmatrix}$ for all l; m ∈ A and r; s ∈ B with l ≥ s; m ≤ r. **(25)** If there is $(l_0 ; r_0) \in (A \times B) \cup (B \times A)$

such that $l_0 \leq F(l_0 ; r_0)$, $r_0 \geq (r_0, l_0)$. Then there exist (l; r) ∈ $(A \times B) \cup (B \times A)$ such that F: $(A \times B) \cup (B \times A) \to A \cup B$ has F (l; r) = l; and F (r; l) = r.

Example 2.7: Let A = $\{U_m(R)/U_m(R)$ is upper triangular matrices over R} and

B = $\{L_m(R)/L_m(R)$ is lower triangular matrices over R} with the bipolar metri

d (P,Q) = $\sum_{I,j=1}^{m} |p_{ij}\text{-}q_{ij}|$ for all P = $(p_{ij})_{m\times m} \in U_m(R)$ and Q = $(q_{ij})_{m\times m} \in L_m(R)$. On the set (A, B), we consider the following relation: (P, Q) ∈ $A^2 \cup B^2$, P ≤ Q ⇔$p_{ij} \leq q_{ij}$ where ≤ is usual ordering. Then clearly, (A, B, d) is a complete bipolar metric space and (A, B, ≤) is a partially ordered set. And (A, B) has the property as in Theorem (2.2). Let F: $(A^2,\ B^2) \rightrightarrows$ (A, B) be defined as F (P, Q) =$\left(\frac{p_{ij}+q_{ij}}{5}\right)$ ∀

(P = $(p_{ij})_{m\times m}$, Q= $(q_{ij})_{m\times m}$) ∈ $A^2 \cup B^2$. Then obviously, F has the mixed monotone property, also there exist $P_0 = (O_{ij})_{m\times m}$ and $Q_0 = (I_{ij})_{m\times m}$ such that

$$F\left((O_{ij})_{m\times m}, (I_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} \geq (O_{ij})_{m\times m}$$

and

$$F\left((I_{ij})_{m\times m}, (O_{ij})_{m\times m}\right) = \left(\frac{O_{ij}+I_{ij}}{5}\right)_{m\times m} \leq (I_{ij})_{m\times m}$$

Taking (P = $(p_{ij})_{m\times m}$, Q= $(q_{ij})_{m\times m}$),

(R = $(r_{ij})_{m\times m}$, S= $(s_{ij})_{m\times m}$) ∈ $A^2 \cup B^2$ with P≥ R and Q≤ S , $p_{ij} \geq r_{ij}$, $q_{ij} \leq s_{ij}$, we have

$$d(F(P, Q), F(R, S)) = d\left(\frac{p_{ij}+q_{ij}}{5}, \frac{r_{ij}+s_{ij}}{5}\right)$$

$$= \frac{1}{5}\sum_{i,j=1}^{m} |(p_{ij}+q_{ij})\text{-}(r_{ij}+s_{ij})|$$

$$\leq \frac{1}{5}\left(\sum_{i,j=1}^{m} |p_{ij}\text{-}r_{ij}| + \sum_{i,j=1}^{m} |q_{ij}\text{-}s_{ij}|\right)$$

$$\leq \frac{1}{5}(d(P,R) + d(Q,S))$$

Therefore, all the conditions of Corollary 1 holds and $(0_{m\times m}, 0_{m\times m})$ is the coupled fixed point of F.

3 CONCLUSIONS

This paper presents some coupled fixed point results by using weak contractive conditions defined on bipolar metric space endowed with partial order and suitable examples that supports the main results.

4 DECLARATION

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